ON THE INTERCHANGE OF LIMIT AND LEBESGUE INTEGRAL FOR A SEQUENCE OF FUNCTIONS*

BY

T. H. HILDEBRANDT

R. L. Jeffery in a paper on *The integrability of a sequence of functions*† has given a number of necessary and sufficient conditions for $\lim_n \int_E f_n = \int_E F$, where f_n and F are summable on the measurable set E and $\lim_n f_n = F$ on E. The object of this note is to give an additional condition for the validity of this interchange, embodied in the

THEOREM. *If $U(n, \delta)$ is the least upper bound, and $L(n, \delta)$ is the greatest lower bound of $\int_e f_n$ for all measurable subsets e of E for which $me \leq \delta$, then a necessary and sufficient condition that $\lim_n \int f_n = \int F$ on E, is that*

$$\lim_{\delta\to 0} \lim_{n\to\infty} [U(n, \delta) + L(n, \delta)] = 0.$$

We are using the notation $\lim_\delta \lim_n$ in the standard sense of the common value of $\lim_\delta \underline{\lim}_n$ and $\lim_\delta \overline{\lim}_n$. In (ϵ, δ) form the condition of the theorem is equivalent to the following:‡ For every $\epsilon > 0$ there exists a δ_ϵ such that for every $\delta \leq \delta_\epsilon$, there exists an $n_{\delta\epsilon}$ such that if $n \geq n_{\delta\epsilon}$, then $|U(n, \delta) + L(n, \delta)| \leq \epsilon$.

We establish the equivalence of our condition with the necessary and sufficient condition I of Jeffery, viz.

$$\lim_{l\to\infty} \int_{C(l,\eta)} f_n = 0 \qquad (n \geq l)$$

where $C(l, \eta)$ is the complement relative to E of the set for which

$$|f_n - f| \leq \eta$$

for every $n \geq l$.

By the definition of $U(n, \delta)$, for every ϵ and δ, there exists a subset e of E of measure less than δ such that

$$U(n, \delta) - \epsilon \leq \int_e f_n \leq U(n, \delta).$$

Since $U(n, \delta) \geq 0$, we can obviously assume that $f_n \geq 0$ on e. Now for any l and η

* Presented to the Society, December 31, 1930; received by the editors December 1, 1930.

† Cf. the present number of these Transactions.

‡ Cf., for instance, Hildebrandt, *Note on interchange of limits*, Bulletin of the American Mathematical Society, vol. 34 (1928), p. 80.

$$\left|\int_e f_n - \int_{eC(l,\eta)} f_n\right| \leqq \int_{e-eC(l,\eta)} |f_n - F| + \int_{e-eC(l,\eta)} |F| .$$

As a consequence if we take δ so that $\delta\eta \leqq \epsilon/2$ and so that, for $me \leqq \delta$, it is true that $\int_e |F| < \epsilon/2$, then provided $n \geqq l$

$$\left|\int_e f_n - \int_{eC(l,\eta)} f_n\right| \leqq \epsilon.$$

Let C^+ be the subset of $C(l, \eta)$ for which $f_n \geqq 0$, and C^- the set for which $f_n < 0$. If then l be chosen so that $mC(l, \eta) \leqq \delta$, and $n \geqq l$, then

$$U(n, \delta) - 2\epsilon \leqq \int_{eC(l,\eta)} f_n = \int_{eC^+} f_n \leqq \int_{C^+} f_n \leqq U(n, \delta).$$

In a similar way we show that

$$L(n, \delta) \leqq \int_{C^-} f_n \leqq L(n, \delta) + 2\epsilon.$$

Hence if $\delta \leqq \delta_\epsilon$, and $l \geqq l_{\delta\epsilon}$, with $n \geqq l$, we have

$$\left|\int_{C(l,\eta)} f_n - [U(n, \delta) + L(n, \delta)]\right| \leqq 2\epsilon.$$

From this statement follows the equivalence of the conditions

$$\lim_{l\to\infty} \int_{C(l,\eta)} f_n = 0 \ (n \geqq l) \text{ and } \lim_{\delta\to 0} \lim_{n\to\infty} [U(n, \delta) + L(n, \delta)] = 0.$$

In so far as

$$\lim_{\delta\to 0} [U(n, \delta) + L(n, \delta)] = 0$$

for every n, our condition is equivalent to the equality of the iterated limits $\lim_\delta \lim_n$ and $\lim_n \lim_\delta$ of the function $U(n, \delta) + L(n, \delta)$. Since the existence of the double limit $\lim_{(n,\delta)\to(\infty,0)}$ is sufficient for this interchange it follows that

$$\lim_{(n,\delta)\to(\infty,0)} [U(n, \delta) + L(n, \delta)] = 0$$

is also a sufficient condition for $\lim_n \int_E f_n = \int_E F$. That it is not necessary follows from the example given by Jeffery in §3.

If the integrals $\int f_n$ are equicontinuous on E, i.e., if

$$\lim_{me\to 0} \int_e f_n = 0$$

uniformly in n, then obviously $\lim_\delta[|U(n, \delta)|+|L(n, \delta)|]=0$, uniformly in n, U and L being even taken with respect to any subset e of E. Applying the standard theorem on interchange of iterated limits, we get at once the Vitali theorem for all subsets e of E. On the other hand, if all the functions f_n are positive on E, then $U(n, \delta)$ converges to zero monotonically in δ, while $L(n, \delta)=0$. By applying the following generalization of Dini's Theorem:*

If $\lim_n\lim_\delta U(n, \delta)=\lim_\delta\lim_n U(n, \delta)$, *and* $U(n, \delta)$ *is monotone in* δ, *then* $\lim_\delta U(n, \delta)$ *exists uniformly in* n,

we find the well known result that when $f_n \geqq 0$, equicontinuity of $\int f_n$ is a necessary condition for $\lim_n \int f_n = \int F$ on E.

The theorem of this note is still valid if the convergence of f_n to f on E is convergence in a measure,† i.e., if $D(n,\eta)$ is the set of points of E for which $|f_n - f| > \eta$ then $\lim_n mD(n,\eta)=0$ for each η. Jeffery's Theorem must be altered so that the condition becomes

$$\lim_n \int_{D(n,\eta)} f_n = 0$$

for every η. The proofs for this more general case require only slight changes from those given.

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* Cf. Bulletin of the American Mathematical Society, vol. 21 (1914), p. 113.

† The possibility of this extension was suggested by L. M. Graves.

University of Michigan,
Ann Arbor, Mich.